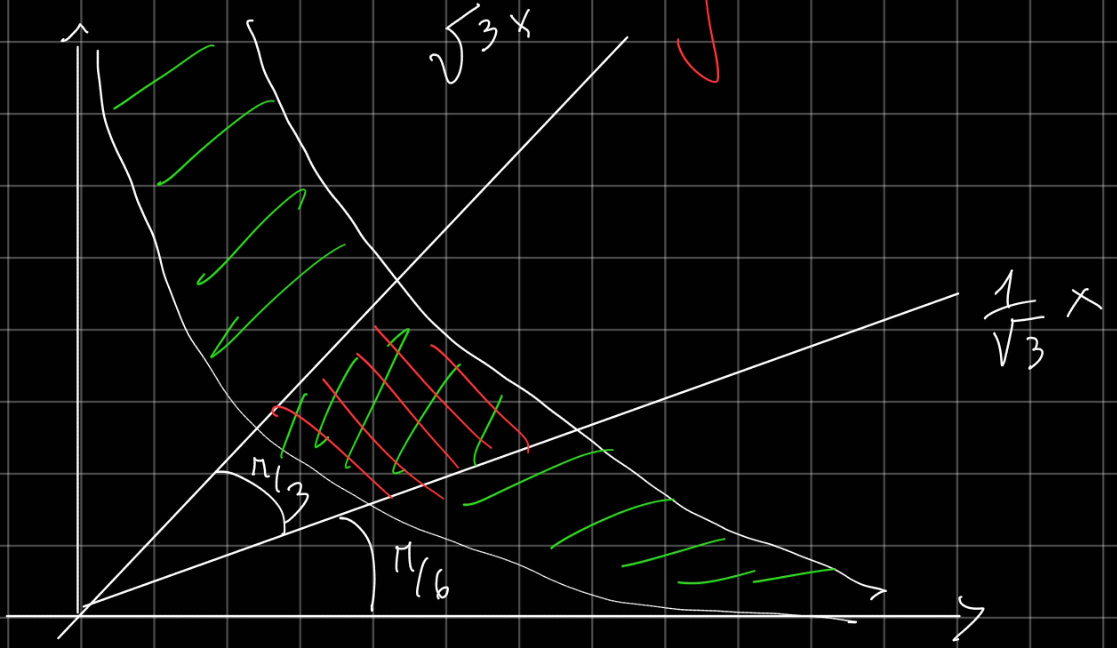
XD

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{1}{\sqrt{3}} x \le y \le \sqrt{3}x,\ x,y \ge 0,\ 1 \le xy \le 2 \right\}$$

$$f : D \to \mathbb{R}$$

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{1}{x} \le y \le \frac{2}{x}$$

Passare alle coordinate polari, ci conviene



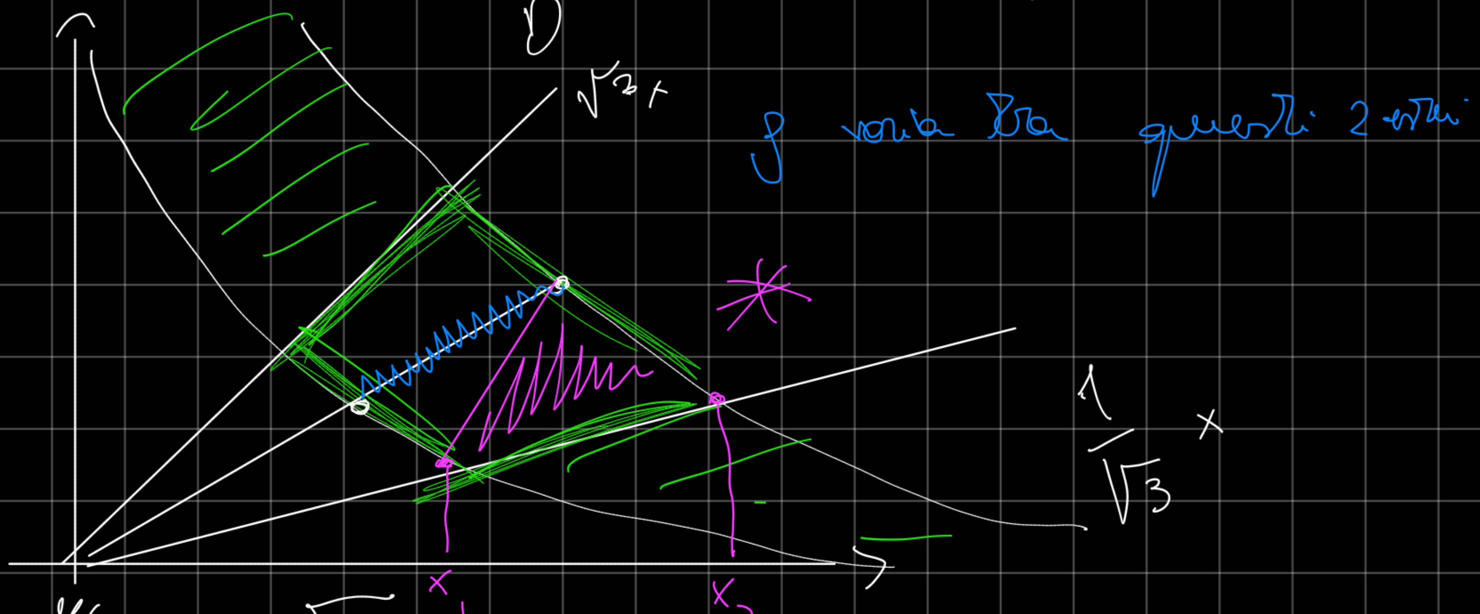
$$1 \le \rho\cos\vartheta \cdot \rho\sin\vartheta \le 2$$

$$1 \le \rho^2 \cos\vartheta \sin\vartheta \le 2$$

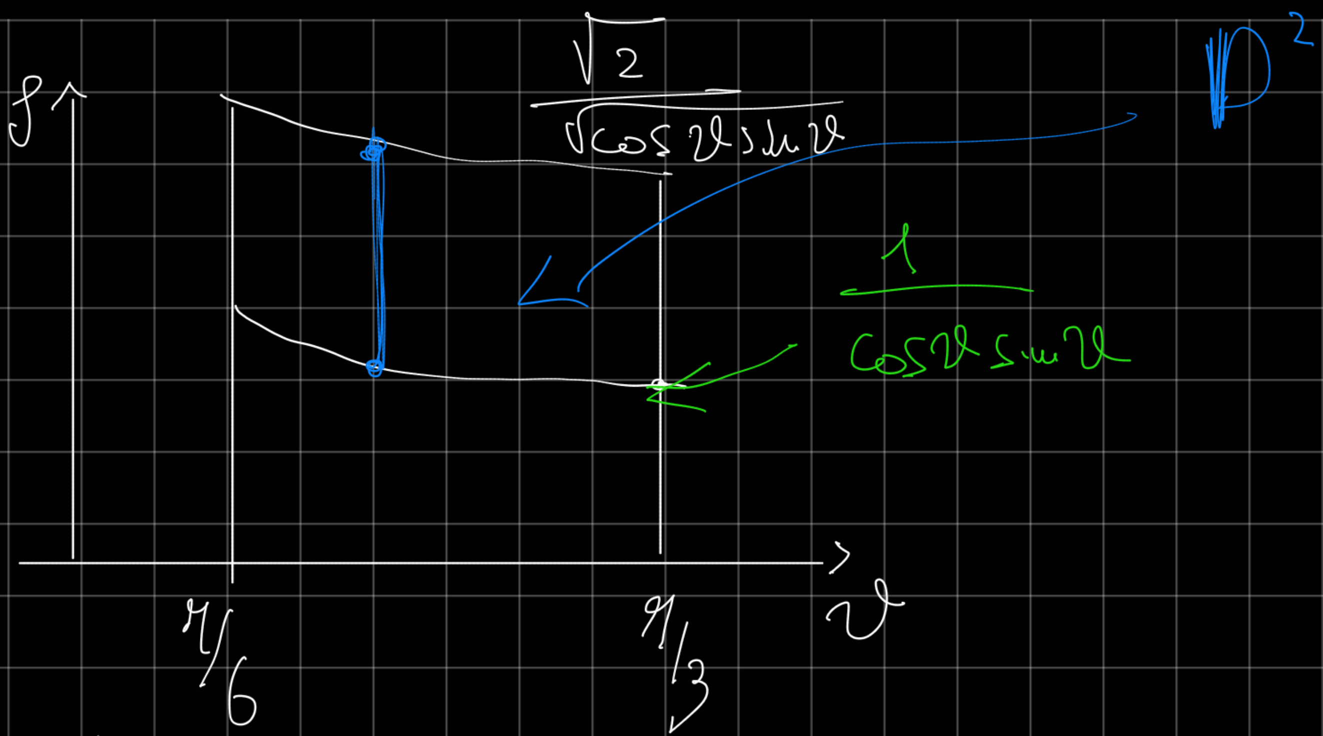
$$\sqrt{\frac{1}{\cos\vartheta\sin\vartheta}} \le \rho \le \sqrt{\frac{2}{\cos\vartheta\sin\vartheta}}$$

$$f(\rho\cos\vartheta, \rho\sin\vartheta) = \frac{\rho\cos\vartheta}{\rho} = \cos\vartheta$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{\tilde{D}} \cos\vartheta \cdot \rho \, d\rho \, d\vartheta$$



$$\int_{\pi/6}^{\pi/3} \left[\int_{\frac{1}{\sqrt{\cos\vartheta\sin\vartheta}}}^{\frac{\sqrt{2}}{\sqrt{\cos\vartheta\sin\vartheta}}} \cos\vartheta \cdot \rho \, d\rho \right] d\vartheta$$



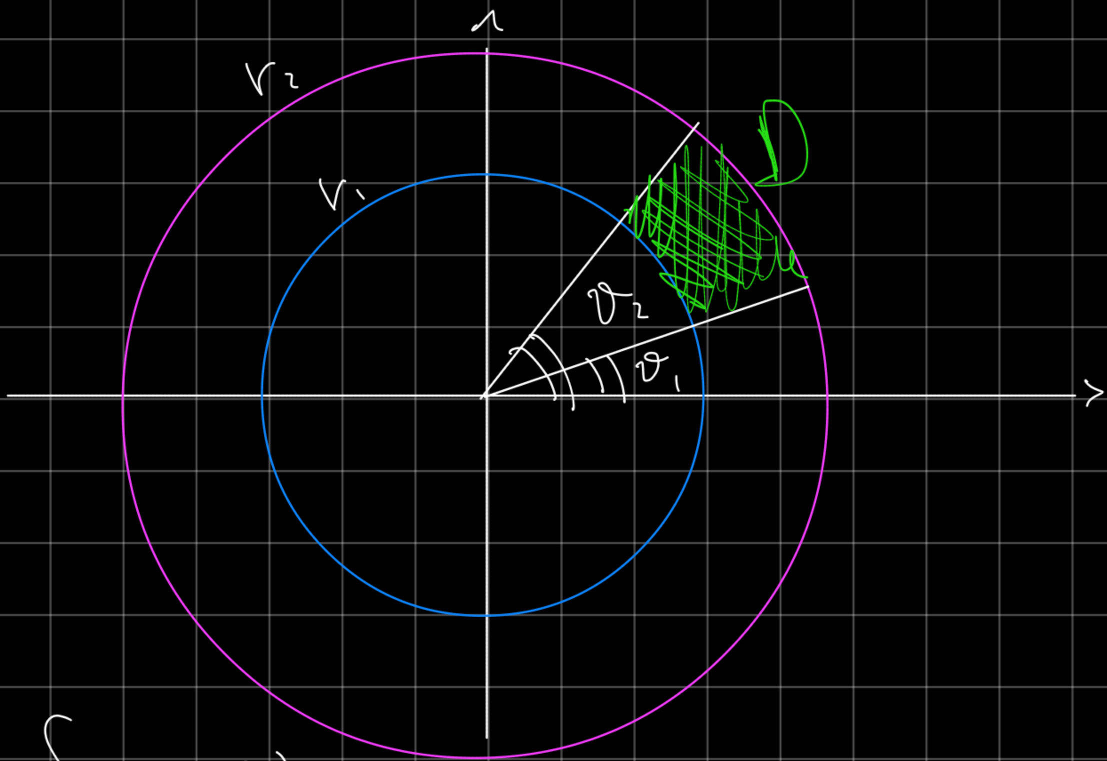
$$\int_{\pi/6}^{\pi/3} \cos\vartheta \left[\frac{\rho^2}{2}\right]_{\sqrt{\frac{1}{\cos\vartheta\sin\vartheta}}}^{\frac{\sqrt{2}}{\cos\vartheta\sin\vartheta}} d\vartheta$$

$$\int_{\pi/3}^{\pi/6} \frac{\cancel{\cos\vartheta}}{2}\,\frac{1}{\cancel{\cos\vartheta}\sin\vartheta}\, d\vartheta = \frac{1}{2}\int_{\pi/3}^{\pi/6} \frac{1}{\sin\vartheta}\, d\vartheta =$$

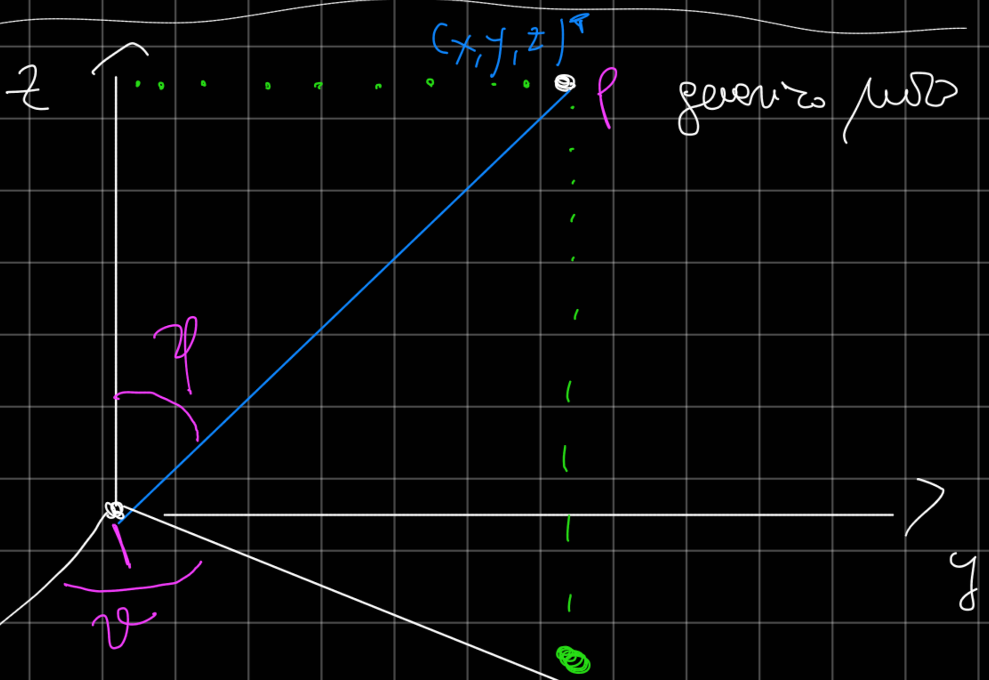
$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} \frac{\sin\vartheta}{1-\cos^2\vartheta}\, d\vartheta = \frac{1}{2}\int_{\underset{1/2}{\cos(\pi/3)}}^{\overset{\sqrt{3}/2}{\cos(\pi/6)}} \frac{1}{1-z^2}\, dz$$

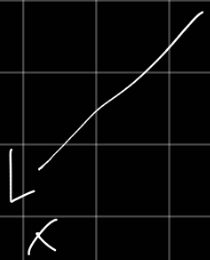
* questo rispetto in cartesiane

$$\int_{x_1}^{x_2}\left[\int_{e/x}^{2/x}\frac{x}{\sqrt{x^2+y^2}}\,dy\right]dx$$



$$\tilde{D}=\{(\rho,\vartheta):\ r_1\le\rho\le r_2\ ,\ \vartheta_1\le\vartheta\le\vartheta_2\}$$





$P_{\pi_{x,y}}$ il proiettato sul piano x,y

distanza dall'origine

$$\rho = \sqrt{x^2+y^2+z^2} \qquad \mathbb{R}^3 \setminus \{(0,0,z)^T : z \in \mathbb{R}\}$$

non voglio che proiettato venga in (0,0)

Proietto il punto sul piano x,y

Scriviamo x,y,z in funzione di queste 3 coord. individuate

(proiettato)

$$\begin{cases} z = \rho \cdot \cos\varphi \\ x = \rho \operatorname{sen}\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \end{cases}$$

$$\begin{cases} \Psi_3(\rho,\varphi,\vartheta) = z = \rho\cos\varphi \\ \Psi_1(\rho,\varphi,\vartheta) = x = \rho\operatorname{sen}\varphi\cos\vartheta \\ \Psi_2(\rho,\varphi,\vartheta) = y = \rho\sin\varphi\sin\vartheta \end{cases}$$

$$\underbrace{(0,+\infty)}_{\text{Dom } \rho} \times \underbrace{(0,\pi)}_{\text{Dom } \varphi} \times \underbrace{[0,2\pi)}_{\text{Dom } \vartheta} \longrightarrow \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$$

$$D\psi_{(\rho,\varphi,\vartheta)} = \begin{pmatrix} \sin\varphi\cos\vartheta & \rho\cos\varphi\cos\vartheta & -\rho\sin\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta & \rho\cos\varphi\sin\vartheta & \rho\sin\varphi\cos\vartheta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix}$$

Determinante, LAPLAS ultima riga.

$$\cancel{(-1)^{3+1}}\cos\varphi\cdot\left(\rho^2\cos\varphi\sin\varphi\cos^2\vartheta - \rho^2\cos\varphi\sin\varphi - \sin^2\vartheta\right)$$

$$+\cancel{(-1)^{3+2}}\cdot(\cancel{+}\rho\sin\varphi)\left(\rho\sin^2\varphi\cos^2\vartheta - \rho\sin^2\varphi\cdot\sin^2\vartheta\right)$$

messo in evidenza

$$= \cos\varphi\left(\rho^2\sin\varphi\cos\varphi\left(\cos^2\vartheta+\sin^2\vartheta\right)\right] + (\rho\sin\varphi)\cdot\rho\sin^2\varphi\left(\cos^2\vartheta+\sin^2\vartheta\right)$$

$$= \cos^2\varphi\cdot\rho^2\sin\varphi + \rho^2\sin^3\varphi = \rho^2\sin\varphi\cdot\left(\cos^2\varphi+\sin^2\varphi\right) = \overset{1}{\tilde{}}$$

$$= \rho^2\sin\varphi \Leftarrow \det\tilde{D}$$

$$\iiint f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{\tilde{D}} f(\rho\sin\varphi\cos\vartheta, \rho\sin\varphi\sin\vartheta, \rho\cos\varphi)\cdot\rho^2\sin\varphi\; d\rho\,d\varphi\,d\vartheta$$

$$\alpha > 0$$

$$D = \left\{(x,y,z)\in\mathbb{R}^3 : 1 \le x^2+y^2+z^2 \le 4,\quad z \ge \sqrt{x^2+y^2}\right\}$$

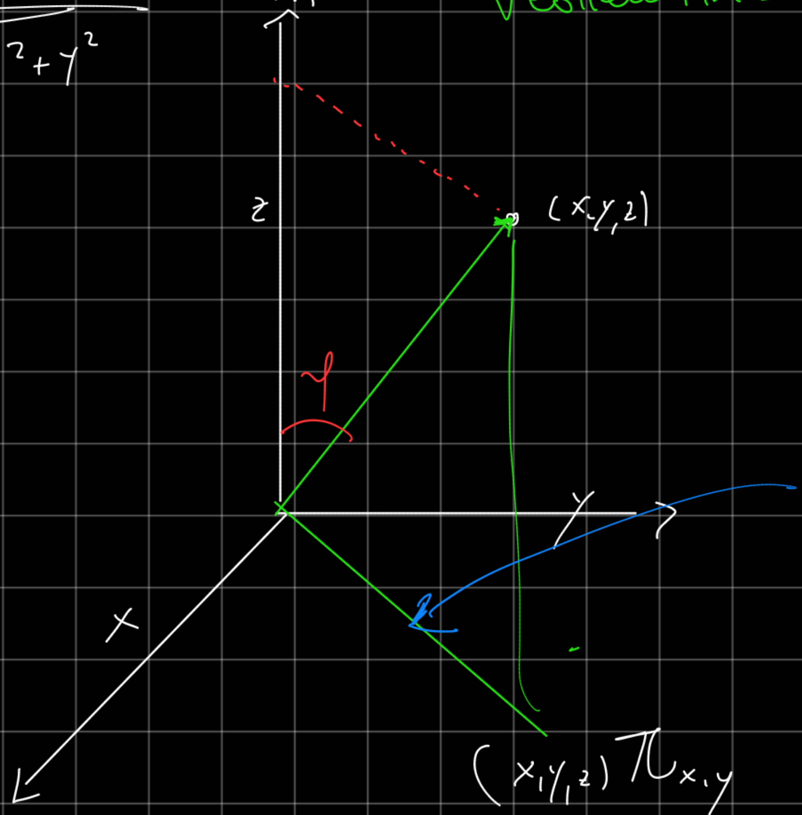
$$f(x,y,z) = \left(x^2+y^2+z^2\right)^{\alpha}$$

$z = m\sqrt{x^2+y^2}$ $\qquad m \neq 0$

$(x,y) \neq (0,0)$

$$\frac{z}{\sqrt{x^2+y^2}} = m$$

Vediamo in 3d cosa significa ciò



$$\cot\varphi = \frac{z}{\sqrt{x^2+y^2}} = m$$

$$\tan\varphi = \frac{\sqrt{x^2+y^2}}{z}$$

$\sqrt{x^2+y^2}$

$$\operatorname{arcctan}\left(\frac{1}{m}\right) = \psi$$

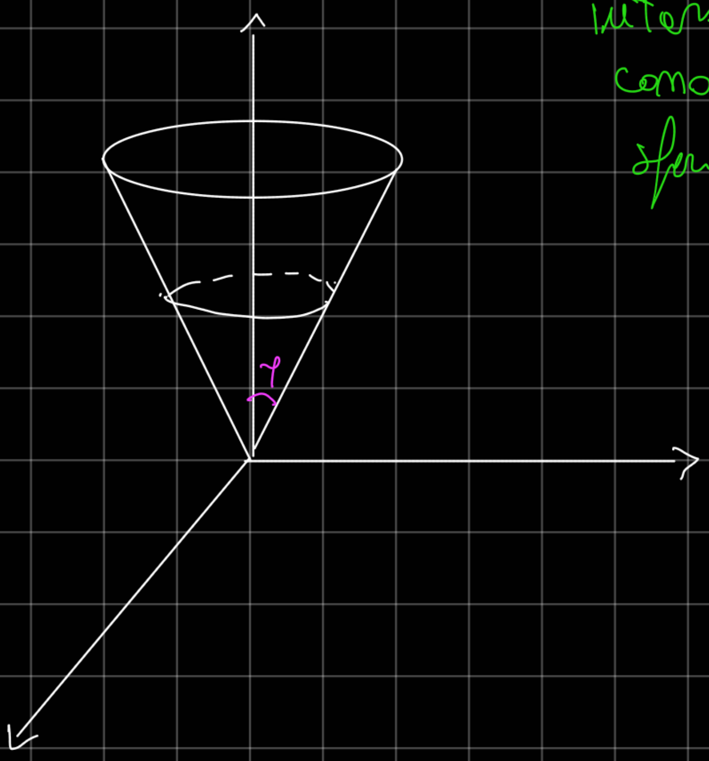
$\varphi \in (0,\pi)$

$$\cot\varphi = \frac{z}{\sqrt{x^2+y^2}} = m \Rightarrow \varphi = \operatorname{arccotan}(m)$$

$\psi = \pi/2 - \varphi$

$\psi = \arctan(m)$

$$\arctan\varphi = \frac{1}{m}$$



interseco questo cono con la mia sfera

$1 \le \rho \le 2 \quad , \quad 0 \le \varphi \le \pi/4 \quad , \quad 0 \le \vartheta < 2\pi$

$$\int_0^{2\pi} \left[\int_0^{\pi/4} \left[\int_1^2 \rho^{2\alpha} \cdot \overset{\text{fattore moltiplic.}}{\underline{\rho^2 \sin\varphi}} \, d\rho \right] d\varphi \right] d\vartheta$$

OSS.

Sia $P = I \times J \times K$ parallelepipedo di $\mathbb{R}^3$

$f(x,y,z) = g_1(x) \cdot g_2(y) \cdot g_3(z)$

Con $g_1 \in R(I)$, $g_2 \in R(J)$, $g_3 \in R(K)$

Allora $f \in R(P)$ e $\iiint_P f = \left(\int_I g_1\right) \cdot \left(\int_J g_2\right) \cdot \left(\int_K g_3\right)$

Supponendo che il dominio sia un PARALLELEPIPEDO

$$= \int_0^{2\pi} \left[\int_0^{\pi/4} \left[\int_1^2 \rho^{2\alpha+2} \sin\varphi \, d\rho \right] d\varphi \right] d\vartheta =$$

$$= \left(\int_0^{2\pi} 1 \, d\vartheta \right) \cdot \left(\int_0^{\pi/4} \sin\varphi \, d\varphi \right) \cdot \left(\int_1^2 \rho^{2\alpha+2} \, d\rho \right) =$$

$$= 2\pi \cdot \left(1 - \frac{\sqrt{2}}{2} \right) \cdot \left(\frac{2^{2\alpha+3} - 1}{2\alpha+3} \right)$$

Passiamo alle coord. cilindriche

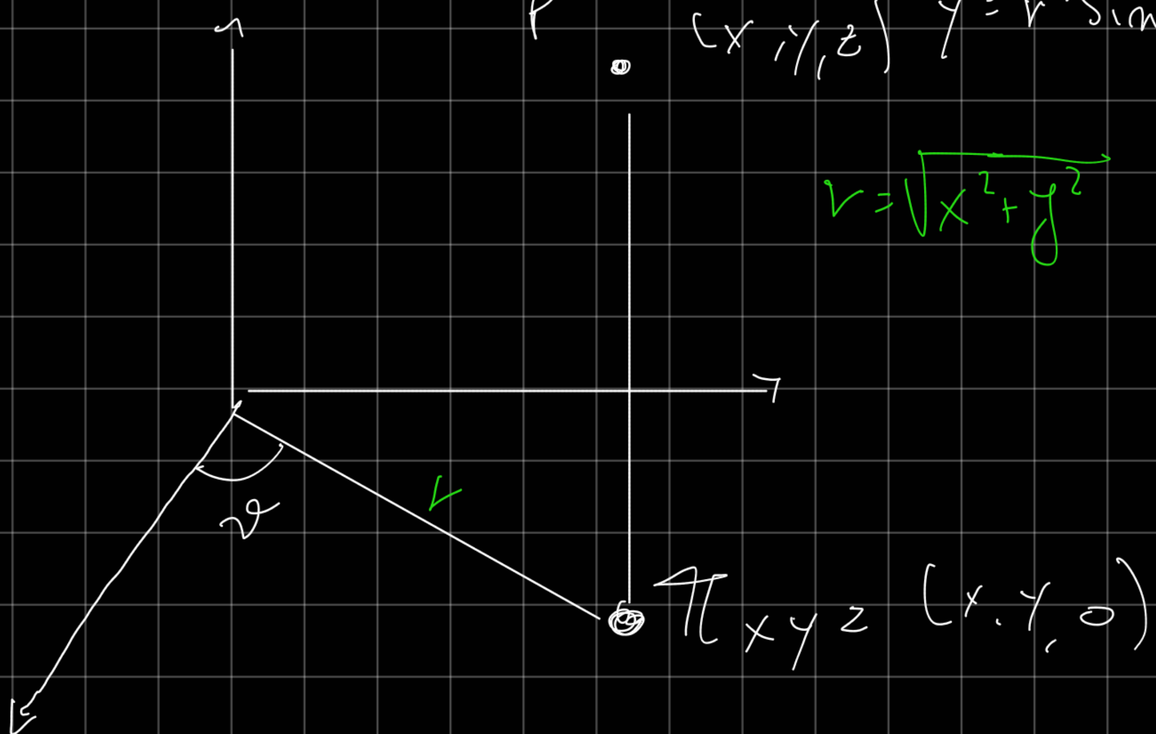
$$\mathbb{R}^3 \setminus \{ (0,0,z) : z \in \mathbb{R} \}$$

$z = z$

$x = r \cdot \cos\vartheta$

$y = r \cdot \sin\vartheta$

$r = \sqrt{x^2 + y^2}$



$$\psi : (0,+\infty) \times [0, 2\pi) \times \mathbb{R}$$

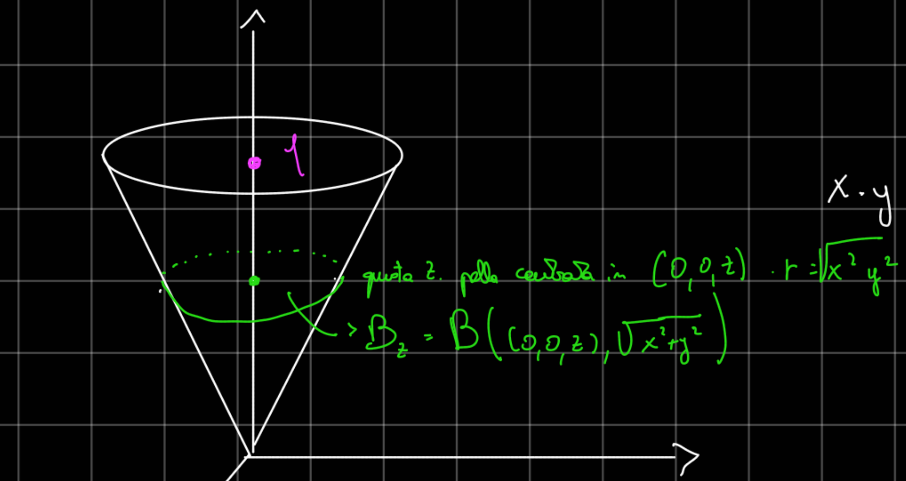
$$(r, \vartheta, z) \longrightarrow (r\cos\vartheta, r\sin\vartheta, z)$$

$$D\psi_{(r,\vartheta,z)} = \begin{pmatrix} \cos\vartheta & -r\sin\vartheta & 0 \\ \sin\vartheta & r\cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(-1)^{3+3} \cdot 1 \cdot r(\cos^2\vartheta + \sin^2\vartheta) = r$$

det. fattore magn.

$$D = \{ (x,y,z) : 1 \geq z \geq \sqrt{x^2+y^2}, y, x \geq 0 \}$$



fatt. magn.

$$\int_0^1 \left[\iint_{B_z} r\cos\vartheta \, r\sin \, \underline{r} \, dr \, d\vartheta \right] dz$$

$$\int_0^1 \left[\iint_{\tilde{B}_z} r^3 \cos\vartheta \sin\vartheta \, dr \, d\vartheta \right] dz =$$

$\sim$ = trasformato in coord. cilindriche

$$\tilde{B}_z = \left\{ (r, \vartheta, z) : 0 \le \vartheta < 2\pi, \ 0 < r \le z \right\}$$

come se fosse in polari

$$= \int_0^1 \left[\int_0^{2\pi} \left[\int_0^z r^3 \cos\vartheta \sin\vartheta \, dr \right] d\vartheta \right] dz$$

sono parametri, posso portarli fuori

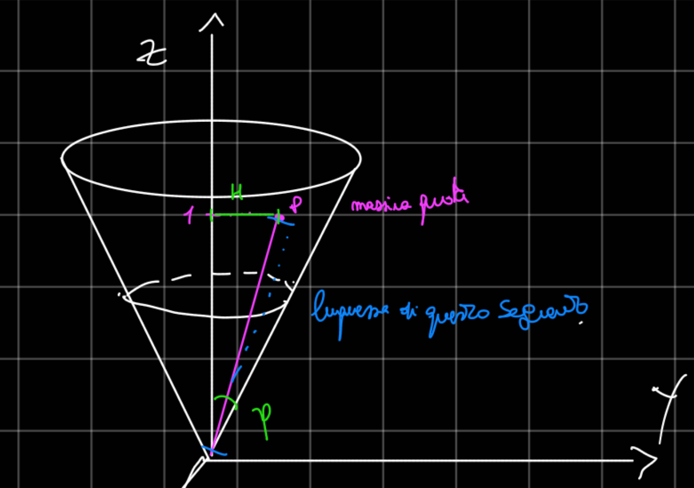
massimo raggio = z

$$\int_0^1 \left[\int_0^{\pi/2} \left[\cos\vartheta \sin\vartheta \int_0^z r^3 \, dr \right] d\vartheta \right] dz =$$

ho ristretto il dominio

$$\int_0^1 \frac{z^4}{4} \cdot \int_0^{\pi/2} \cos\vartheta \sin\vartheta \, d\vartheta \Big] dz =$$

$$\int_0^1 \frac{z^4}{4} \cdot \frac{1}{2} \cdot 1 \, dz = \frac{1}{8} \cdot \frac{1}{5} = \frac{1}{40}$$



$$\cos\varphi = \frac{\text{cateto adiacente}}{\text{ipotenusa}}$$

$$\frac{1 = \overline{OH}}{\overline{OP}}$$

$$\frac{1}{\overline{OP}} = \cos\varphi$$

$$\overline{OP} = \frac{1}{\cos\varphi}$$

$$\int_0^{\pi/2}\left[\int_0^{\pi/4}\left[\int_0^{\frac{1}{\cos\varphi}} \rho\sin\varphi\cos\vartheta\cdot\rho\sin\varphi\sin\vartheta\cdot\underbrace{\rho^2\sin\varphi}_{}\,d\rho\right]d\varphi\right]d\vartheta$$

fatt. magg.

$$\int_0^{\pi/2}\left[\int_0^{\pi/4}\left[\int_0^{\frac{1}{\cos\varphi}} \rho^4\sin^3\varphi\cos\vartheta\cdot\cancel{\rho\sin\varphi}\sin\vartheta\cdot\cancel{\rho^2\sin\varphi}\,d\rho\right]d\varphi\right]d\vartheta$$

fatt. magg.

$$\int_0^{\pi/2}\left[\int_0^{\pi/4}\sin^3\varphi\cos\vartheta\sin\vartheta\cdot\frac{1}{5}\,\frac{1}{\cos^5\varphi}\,d\varphi\right]d\vartheta$$

ci siamo ridotti ad integrare in un rettangolo

$$=\left[\int_0^{\pi/2}\cos\vartheta\sin\vartheta\,d\vartheta\right]\cdot\int_0^{\pi/4}\frac{\sin^3\varphi}{\cos^5\varphi}\,d\varphi$$